# Tutorial 5

## Exercise 1

Find bases and dimensions for the null space of the matrix:

(i) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

→ Need to write in reduced row echelon form;

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{matrix} \\ r_2 - r_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ r_2 \times -1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} r_1 - 2r_2 \\ r_2 \end{matrix}$$

$\Rightarrow \tilde{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ [row equivalent matrix in reduced row echelon form]

→ Now we can compute the null space:

$$\tilde{A}\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$x_1 = x_3$$
$$x_2 = -x_3$$

write $x_3 = t$ [free variable]

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

null space basis of A: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

→ dimension of the null space is 1 as there is 1 vector in the null space basis.

(ii) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \\ -4 & -8 \\ -1 & -2 \end{pmatrix}$

→ write in reduced row echelon form.

$$\begin{array}{c} \begin{pmatrix} 3 & 6 \\ 1 & 2 \\ -4 & -8 \\ -1 & -2 \end{pmatrix} \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}$$

swap row 2 + row 1

$$\begin{array}{c} \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ -4 & -8 \\ -1 & -2 \end{pmatrix} \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -4 & -8 \\ -1 & -2 \end{pmatrix} \quad r_2 - 3r_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ -1 & -2 \end{pmatrix} \quad r_3 + 4r_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad r_4 + r_1$$

↳ row equivalent matrix in reduced row echelon form.

$$\tilde{A}\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 0 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$\Rightarrow x_1 = -2x_2$

let $x_2 = t$ [free variable]

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\therefore$ null space basis of $A$ : $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

dimension is 1
(one vector in the null space basis)

Exercise 2

Find a subset of the vectors that forms a basis of their span:

(i) $\vec{v}_1 = (-1, 1, -2)$

$\vec{v}_2 = (2, -2, 4)$

Form the matrix A with these vectors as column vectors and transform A to a matrix $\tilde{A}$ in row echelon form.

$$A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \\ -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \\ -2 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \tilde{A}$$

→ Column 1 has a leading 1 in it.

Therefore $\boxed{\vec{v}_1}$ forms a basis of their span.

(ii) $\vec{v}_1 = (2, 1)$

$\vec{v}_2 = (1, 2)$

$\vec{v}_3 = (-1, 1)$

$\vec{v}_4 = (-1, 2)$

$$A = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & -1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -3 & -5 \end{pmatrix} \begin{matrix} r_1 \\ r_2 - 2r_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 5/3 \end{pmatrix}$$ Leading 1s

$\Rightarrow$ $\vec{v_1}$ and $\vec{v_2}$ form a basis of the span of these 4 vectors.

## Exercise 3

Find the rank and nullity of the matrix:

(i) $$\begin{pmatrix} -2 & 2 & -4 \\ 3 & 3 & 6 \end{pmatrix}$$

Rank = dimension of each row and column space.

$\rightarrow$ compute the rank of a matrix by counting leading 1s in any row echelon form matrix $\tilde{A}$ that is row equivalent to A.

$$\begin{pmatrix} -2 & 2 & -4 \\ 3 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 6 & 0 \end{pmatrix} \quad r_2 - 3r_1$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$ 2 leading 1s $\Rightarrow$ $\boxed{\text{rank} = 2}$

nullity = dimension of row of A − rank A

nullity = 3 − 2 = $\boxed{1}$

(ii) $\begin{pmatrix} 4 & 3 & -6 \\ 1 & -1 & 2 \\ -11 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 2 \\ 4 & 3 & -6 \\ -11 & 0 & 0 \end{pmatrix}$ swap $r_1$ + $r_2$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 7 & -14 \\ 1 & 0 & 0 \end{pmatrix}$ $\begin{matrix} r_1 \\ r_2 - 4r_1 \\ r_3 / -11 \end{matrix}$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 7 & -14 \\ 0 & 1 & -2 \end{pmatrix}$ $\begin{matrix} r_1 \\ r_2 \\ r_3 - r_1 \end{matrix}$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$ $r_2 / 7$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $r_3 - r_2$

2 leading 1s $\Rightarrow$ $\boxed{\text{rank} = 2}$

nullity = dimension of row of A − rank A

nullity = $3 - 2 = \boxed{1}$